Analysis of Globally Stable Periodic Orbits in Permutation Elementary Cellular Automata

Taiji Okano, Mikito Onuki and Toshimichi Saito
EE Dept., HOSEI University, Koganei, Tokyo, 184-8584 Japan
taiji.okano.7e@stu.hosei.ac.jp, mikito.onuki.8v@stu.hosei.ac.jp, tsaito@hosei.ac.jp

***Abstract*—We consider a simple three-layer dynamical systems related to recurrent neural networks. The input to hidden layers construct an elementary cellular automaton and the hidden to output layers are one-to-one connection described by a permutation. Depending on the permutation connections, the network can generate various periodic orbits of binary vectors. Especially, we have discovered globally stable periodic orbits such that almost all initial points fall into the orbits. Based on numerical analysis, we present an important conjecture for property of globally stable periodic orbits. This is a first step to consider various periodic orbits and their engineering applications.**

***Index Terms*—recurrent neural networks, elementary cellular automata, periodic orbits, stability,**

I. Introduction

Discrete-time recurrent neural networks ([1]-[3]) are characterized by nonlinear activation function and real valued connection parameters. The dynamics is described by an autonomous difference equation of real state variables. Depending on the parameters, the networks can exhibit various nonlinear phenomena: multiple fixed points, multiple periodic orbits, chaos, and related bifurcation. The real/potential applications include combinatorial optimization problems solvers [3], associative memories [1], and reservoir computing [4]. The networks are important subject in analysis of nonlinear dynamics and engineering applications. In the networks, multiple fixed points have been analyzed sufficiently and the results have contributed to develop applications [1] [3]. However, analysis of periodic orbits is difficult because of complicated dynamics for an enormous number of parameters. If periodic orbits are analyzed sufficiently, the results contribute to further understanding of network dynamics and real-world applications.

In this article, we consider periodic orbits in a simple three-layer dynamical system related to the DT-RNNs: a permutation elementary cellular automaton (PECA). In the PECA, the input to hidden layers construct an elementary cellular automaton (ECA [5] [6]) whose dynamics is governed by rules of Boolean functions from three inputs to one output. The hidden to output layers are one-to-one connection described by a permutation [7]-[9]. The number of parameters is much smaller that that of usual DT-RNNs, The PECA is well suited for precise analysis of BPOs and FPGA based hardware implementation for engineering applications.

As permutation connection varies, the PECA can generate various periodic orbits of binary vectors (BPOs) that are impossible in the ECAs. Especially, we have discovered globally stable binary periodic orbits (GBPOs) such that almost all initial points fall into the GBPOs. Real/potential applications of the GBPOs/BPOs include control signals of switching power converters [10]-[12], control signals of walking robots [13]-[15], and approximation signals of time-series [4]. Since the GBPOs/BPOs are stabile, the control/approximation signals are robust against disturbance. After trial-and-errors, we present an important conjecture for property of GBPOs. This is a first step to realize precise analysis of various BPOs and their engineering applications.

II. Elementary Cellular Automata

First, we introduce ECAs on a ring of N cells. The dynamics is described by

$$x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t), \quad i \in \{1, \cdots, N\},\ N \geq 3 \tag{1}$$

where $x_i^t \in \{0,1\}$ is the i-th binary state variable at discrete time t $x_0^t \equiv x_N^t$ and $x_{N+1}^t \equiv x_1^t$ for the ring topology. A Boolean function f transforms three binary inputs to one binary output. For example,

$$\begin{array}{llll} f(0,0,0)=0, & f(0,0,1)=1, & f(0,1,0)=0, & f(0,1,1)=1 \\ f(1,0,0)=1, & f(1,0,1)=0, & f(1,1,0)=1, & f(1,1,1)=0 \end{array}$$

Decimal expression of the 8 outputs is referred to as the rule number (RN). In this example, $(01011010)_2 = 90_{10}$ gives RN90. Fig. 1 shows ECA of RN90 and a BPO with period 7.

III. Permutation Elementary Cellular Automata

Applying permutation connection to the ECA, the PECA is constructed. The dynamics is described by the following autonomous difference equation of binary state variables.

$$\begin{array}{l} x_i^{t+1} = y_{\sigma(i)}^t, y_i^t = F(x_{i-1}^t, x_i^t, x_{i+1}^t) \\ \sigma = \begin{pmatrix} 1 & 2 & \cdots & N \\ \sigma(1) & \sigma(2) & \cdots & \sigma(N) \end{pmatrix} \end{array} \tag{2}$$

where $y_i^t \in \boldsymbol{B}$ is the i-th binary hidden state and σ is a permutation. As shown in Fig. 2, the input to hidden layers construct an ECA and the hidden and output layers are connected by a permutation. The permutation connection transforms the binary hidden state vector $\boldsymbol{y}^t$ into the binary output vector $\boldsymbol{x}^{t+1}$. For convenience, Eq. (2) is abbreviated by $\boldsymbol{x}^{t+1} = F(\boldsymbol{x}^t)$. $\boldsymbol{x}^t \equiv (x_1^t, \cdots, x_N^t) \in \boldsymbol{B}^N$ where $\boldsymbol{B}^N$ is the set of all N-dimensional binary vectors. The PECA

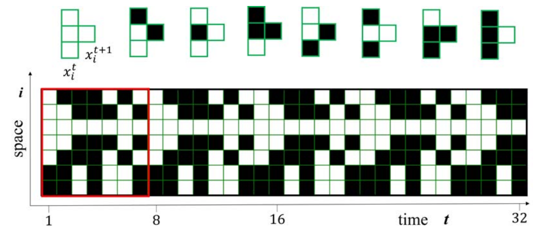

Fig. 1. ECA and BPOs with period 7 for RN90. The white and black boxes denote $x_i^t = 0$ and $x_i^t = 1$, respectively.

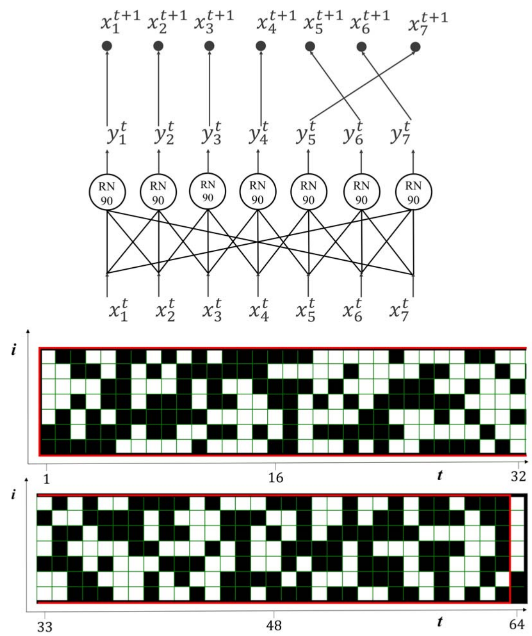

Fig. 2. PECA and GBPO with period 63 for RN90 and P1234675.

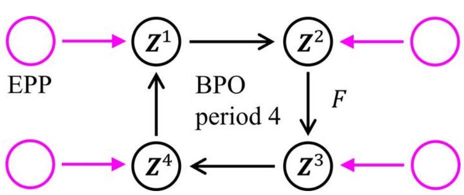

Fig. 3. Binary periodic orbit (BPO) and eventually periodic points (EPPs)

is characterized by the rule number (RN) and the permutation identifier P$\sigma(1)\cdots\sigma(N)$ (PID). The RN and PID are regarded as parameters of the PECA. We give fundamental definition.

Definition: A point $\boldsymbol{z}^p \in B^N$ is said to be a binary periodic point (BPP) with period p if $F^p(\boldsymbol{z}) = \boldsymbol{z}_p$ and $F(\boldsymbol{z}_p)$ to $F^p(\boldsymbol{z}_p)$ are all different where F^k is the k-fold composition of F. A sequence of the BPPs, $\{F(\boldsymbol{z}_p), \cdots, F^p(\boldsymbol{z}_p)\}$, is said to be a BPO with period p. A point $\boldsymbol{z}_e$ is said to be an eventually periodic point (EPP) if $\boldsymbol{z}_e$ is not a BPP but falls into the BPO (see Fig. 3). Let q denote the number of EPPs to a BPO. As q increases, stability of the BPO becomes stronger.

We have investigated various BPOs and a typical example is shown in Fig. 2: a 7-dimensional BPO with period $p = 63$ and $q = 63$. In the PECA, two end points (all 0 and all 1) are either fixed points or periodic points with period 2. Except for the two end points, half of $(2^7 - 2 = 126)$ elements constructs the BPO and the other half elements are EPPs to the BPO. Such a BPO is referred to as a globally stable binary periodic orbit (GBPO). The GBPO is an important example in analysis of BPOs and is useful in application because of the strong stability and long period. The ECAs cannot generate GBPOs. In order to realize analysis of various BPOs and their applications, now we are trying to prove the following.

Conjecture: Let dimension N be odd and let the rule number be RN90. As PID varies, the PECA generates various BPOs where period p and the number of EPPs $q = p$. The BPOs include the GBPO with $p = q = (2^N - 2)/2$.

IV. Conclusions

The PECAs are introduced and typical GBPOs are demonstrated in this article. Adjusting global permutation connection, the PECA generate various BPOs. After trial-and-errors, we have presented an important conjecture for the GBPOs. Future problems are many, including the following: (1) Proof of the conjecture. (2) Mechanism to reinforce stability of GBPOs. (3) Real/potential engineering applications of GBPOs such as robust control signals of switching circuits and robust approximation signal of time-series.

References

[1] J. J. Hopfield, Neural networks and physical systems with emergent collective computation abilities, Proc. Nat. Acad. Sci. 79, pp. 2554-2558, 1982.

[2] M. Adachi and K. Aihara, Associative dynamics in a chaotic neural network, Neural Networks, 10, 1, pp. 83-98, 1997.

[3] J. J. Hopfield and D. W. Tank, 'Neural' computation of decisions optimization problems, Biological Cybern., 52, pp. 141-152, 1985.

[4] G. Tanaka, T. Yamane, J. B. Héroux, R. Nakane, N. Kanazawa, S. Takeda, H. Numata, D. Nakano, and A. Hirose, Recent advances in physical reservoir computing: a review. Neural Networks, 115, pp. 100-123, 2019.

[5] S. Wolfram, Cellular Automata and Complexity: Collected Papers, CRC Press, 2018.

[6] L. O. Chua, A nonlinear dynamics perspective of Wolfram's new kind of science, World Scientific, 2006.

[7] T. Okano and T. Saito, Permutation elementary cellular automata: analysis and application of simple examples, M. Tanveer et al. (Eds.): ICONIP 2022, LNCS 13623, pp. 321-330, 2023.

[8] H. Udagawa, T. Okano, and T. Saito, Permutation binary neural networks: analysis of periodic orbits and its applications, Discrete Contin. Dyn. Syst. Ser. B, 28, 1, pp. 748-764, 2023.

[9] M. Onuki, K. Saka and T. Saito, A variety of globally stable periodic orbits in permutation binary neural networks, Discrete Contin. Dyn. Syst. Ser. B, 28, 11, pp. 5800-5813, 2023.

[10] W. Holderbaum, Application of neural network to hybrid systems with binary inputs, IEEE Trans. Neural Netw. 18, 4, pp. 1254-1261, 2007.

[11] P. W. Wheeler, J. Rodriguez, J. C. Clare, L. Empringham, A. Weinstein, Matrix converters: a technology review, IEEE Tran. Ind. Electron., 49, 2, pp. 276-288, 2002.

[12] R. Sato and T. Saito, Stabilization of desired periodic orbits in dynamic binary neural networks, Neurocomputing 248, pp. 19-27, 2017.

[13] P. Ramdya, R. Thandiackal, R. Cherney, T. Asselborn, A. J. Ljspeert, R. Benton, D. Floreano, Climbing favours the tripod gait over alternative faster insect gaits, Nat. Commun. 8, 14494 (2017)

[14] M. Lodi, A. L. Shilnikov, M. Storace, Design principles for central pattern generators with preset rhythms, IEEE Trans. Neural Netw. Learn. Syst. 31(9) (2020) 3658-3669.

[15] T. Suzuki, T. Saito, Synthesis of three-layer dynamic binary neural networks for control of hexapod walking robots, Proc. CNNA, 10.1109/CNNA49188.2021.9610809, 2021.